ISRAEL JOURNAL OF MATHEMATICS, Vol. 57, No. 1, 1987

ON THE DISTRIBUTION OF WEIERSTRASS POINTS ON SINGULAR CURVES

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ABSTRACT

Weierstrass points are defined for invertible sheaves on integral, projective Gorenstein curves. An example is given of a rational nodal curve X and an invertible sheaf $\mathscr{L}$ of positive degree on X such that the set of all higher order Weierstrass points of $\mathscr{L}$ is not dense in X.

1. Introduction

Let Y be a smooth, projective curve of genus $g \geqq 2$ over $\mathbf{C}$ and let Ω denote the canonical bundle on Y. A point $P \in Y$ is called a Weierstrass point (of order 1) if $\dim H^0(Y, \Omega(-gP)) > 0$. For $n > 1$, put

$$\gamma_n = \dim H^0(Y, \Omega^{\otimes n}) = (2n-1)(g-1).$$

A point $P \in Y$ is called a Weierstrass point of order n if

$$\dim H^0(Y, \Omega^{\otimes n}(-\gamma_n P)) > 0.$$

Equivalently, Weierstrass points of order n may be defined in terms of n-gaps or in terms of the Wronskian formed from a basis of $H^0(Y, \Omega^{\otimes n})$ (cf. [3, pp. 84–85]).

More generally, if $\mathscr{L}$ is an invertible sheaf on Y, then one may define P to be a Weierstrass point of order n of $\mathscr{L}$ if $\dim H^0(Y, \mathscr{L}^{\otimes n}(-s_n P)) > 0$, where $s_n = \dim H^0(Y, \mathscr{L}^{\otimes n})$ (cf. [12], [10]). This agrees with the definition in terms of Wronskians (as in [5], [11]). Put

$$W(\mathscr{L}) = \{P \in Y\colon P \text{ is a Weierstrass point of order } n \text{ of } \mathscr{L} \text{ for some } n \geqq 1\}.$$

Suppose $\deg \mathscr{L} > 0$. Then Olsen [12] showed that $W(\mathscr{L})$ is dense (in the complex topology) in Y and Mumford (unpublished, but see [8]) and Neeman [10] established the stronger result that $W(\mathscr{L})$ is uniformly distributed in Y. These

Received Janaury 28, 1986

three proofs all make essential use of the theta divisor on the Jacobian of the curve.

Widland [13] has extended the classical notion of Weierstrass points to integral, projective Gorenstein curves. Singular points are always Weierstrass points and have high Weierstrass weight. This may be interpreted to mean that as a family of smooth curves degenerates to an irreducible Gorenstein curve (e.g., a curve with nodes), many of the Weierstrass points tend toward the singularities. In this regard, Diaz [2] has shown that "the generic non-separating node on a uninodal stable curve is a limit of exactly $(g-1)g$ Weierstrass points on nearby smooth curves."

In the next section, we extend Widland's definition to treat Weierstrass points of order n of an invertible sheaf on an integral, projective Gorenstein curve. It then may seem reasonable to expect that the set of all higher order Weierstrass points of such a line bundle, assuming the bundle has positive degree, will be dense. On the other hand, the fact that many Weierstrass points on "nearby" smooth curves tend toward the singularities might be expected to "skew" the distribution of Weierstrass points. In the final section, we give an explicit example of a rational nodal curve X and an invertible sheaf $\mathscr{L}$ on X of positive degree such that the set of higher order Weierstrass points of $\mathscr{L}$ is not dense in X. Our argument uses the analogue of the theta divisor on the generalized Jacobian of X.

We work over the complex numbers and we will let $\mathbf{P}^1$ denote the complex projective line, which we also identify with the extended complex plane. We will use the terms "invertible sheaf" and "line bundle" interchangeably. We thank Carl Widland and Carruth McGehee for helpful conversations.

2. Weierstrass points of line bundles on Gorenstein curves

Let X be an integral, projective Gorenstein curve of arithmetic genus $g>0$ over $\mathbf{C}$. Let ω denote the canonical bundle of dualizing differentials on X and suppose $\mathscr{L}$ is an invertible sheaf on X. Put $s=\dim H^0(X,\mathscr{L})=h^0(\mathscr{L})$. Assume $s>0$ and choose a basis $\phi_1,\dots,\phi_s$ for $H^0(X,\mathscr{L})$. We will define a section of $\mathscr{L}^{\otimes s}\otimes\omega^{\otimes(s-1)s/2}$, as follows. Suppose that $\{U^{(\alpha)}\}$ is a covering of X by open subsets such that $\mathscr{L}(U^{(\alpha)})$ (resp. $\omega(U^{(\alpha)})$) is a free $\mathcal{O}_X(U^{(\alpha)})$-module generated by $\omega^{(\alpha)}$ (resp. by $\tau^{(\alpha)}$). Define $F_{i,j}^{(\alpha)}\in\Gamma(U^{(\alpha)},\mathcal{O}_X)$ by

$$\phi_j\,\big|_{U^{(\alpha)}}=F_{1,j}^{(\alpha)}\psi^{(\alpha)}\qquad\text{for } j=1,\dots,s,$$

$$dF_{i-1,j}^{(\alpha)}=F_{i,j}^{(\alpha)}\tau^{(\alpha)}\qquad\text{for } i=2,\dots,s \text{ and } j=1,\dots,s.$$

Put

$$\rho^{(\alpha)} = \det[F_{i,j}^{(\alpha)}](\psi^{(\alpha)})^s(\tau^{(\alpha)})^{(s-1)s/2}, \qquad i, j = 1, \ldots, s.$$

It is not hard to see, as in the classical case (cf. [3, p. 85]), that $\rho^{(\alpha)} = \rho^{(\beta)}$ in $U^{(\alpha)} \cap U^{(\beta)}$. Hence the $\rho^{(\alpha)}$ determine a section ρ in $H^0(X, \mathscr{L}^{\otimes s} \otimes \omega^{\otimes(s-1)s/2})$. It is easy to see that a different choice of a basis for $H^0(X, \mathscr{L})$ would result in ρ being multiplied by a nonzero scalar. Therefore, the order of vanishing of ρ at P is independent of the choice of basis of $H^0(X, \mathscr{L})$. By the order of vanishing of ρ at P, we mean the following. If ψ generates $\mathscr{L}_P$ and τ generates ω_P, then we may write

$$\rho = f\psi^s\tau^{(s-1)s/2}.$$

Then

$$\operatorname{ord}_P \rho = \operatorname{ord}_P f = \dim \mathcal{O}/(f) = \dim \tilde{\mathcal{O}}/(f),$$

where $\mathcal{O}$ denotes the local ring at P and $\tilde{\mathcal{O}}$ is its normalization.

Definitions. Suppose that $P \in X$. The *$\mathscr{L}$-Weierstrass weight of P*, denoted $W_{\mathscr{L}}(P)$, is defined to be $\operatorname{ord}_P \rho$. We call P a *Weierstrass point, or W-pt., of $\mathscr{L}$* if $W_{\mathscr{L}}(P) > 0$. We call P a *Weierstrass point of order n of $\mathscr{L}$* if P is a Weierstrass point of $\mathscr{L}^{\otimes n}$.

Remarks. (1) Widland [13] gave this definition for Weierstrass points of the bundle of dualizing differentials.

(2) This definition for the bundle of dualizing differentials is also perhaps implicit in a paper of Arakelov [1].

(3) We would prefer to define W-pts. in terms of singularities of a map between two vector bundles, as in [6], but it is not clear what bundle should replace the sheaf of principal parts on a singular curve. (This sheaf is no longer locally free.)

Proposition 1. *The number of W-pts. of $\mathscr{L}$, counting multiplicities, is* $s \cdot \deg(\mathscr{L}) + (s-1)s(g-1)$.

Proof. This is immediate from the definitions and a calculation of the degree of $\mathscr{L}^{\otimes s} \otimes \omega^{\otimes(s-1)s/2}$. ■

The theory of W-pts. as far as smooth points are concerned is quite similar to the theory on nonsingular curves. At a smooth point, one may define a sequence of gaps and, as in [5], we have

PROPOSITION 2. *Suppose that P is a smooth point of X. Then P is a W-pt. of $\mathcal{L}$ if and only if $h^0(\mathcal{L}^{\otimes n}(-sP)) \neq 0$.*

Put $\delta_P = \dim \tilde{\mathcal{O}}/\mathcal{O}$. We recall that P is singular if and only if $\delta > 0$. Following Widland, we prove

PROPOSITION 3. $W_{\mathcal{L}}(P) \geqq \delta_P \cdot s \cdot (s-1)$.

PROOF. Let $\pi: \tilde{X} \to X$ denote the normalization of X. Suppose that $t \in K(\tilde{X})$ is a rational function such that $\mathrm{ord}_Q\, t = 1$ for all $Q \in \pi^{-1}(P)$ and let h be a generator (in $\tilde{\mathcal{O}}$) of $\mathcal{C}$, the conductor of $\mathcal{O}$ in $\tilde{\mathcal{O}}$. Then $\tau = dt/h$ generates $\omega_{X,P}$.

Let $\phi_1, \ldots, \phi_s$ be a basis for $H^0(X, \mathcal{L})$ and suppose ψ generates $\mathcal{L}_P$. Write $\phi_{j,P} = F_{1,j}\psi$. Then

$$dF_{1,j} = \frac{dF_{1,j}}{dt}\, dt = h \frac{dF_{1,j}}{dt} \frac{dt}{h} .$$

Hence

$$F_{2j} = h \frac{dF_{1,j}}{dt} .$$

Then

$$\begin{aligned} d(F_{2,j}) &= \left(h \frac{d^2 F_{1,j}}{dt^2} + \frac{dh}{dt} \frac{dF_{1,j}}{dt} \right) dt \\ &= \left(h^2 \frac{d^2 F_{1,j}}{dt^2} + \frac{dh}{dt}\, h \frac{dF_{1,j}}{dt} \right) \frac{dt}{h} . \end{aligned}$$

So,

$$F_{3,j} = h^2 \frac{d^2 F_{1,j}}{dt^2} + \frac{dh}{dt} F_{2,j}.$$

Continuing in this manner, we see that we may write

$$F_{i,j} = h^{i-1} \frac{d^{i-1} F_{1,j}}{dt^{i-1}} + \sum_{l=2}^{i-1} g_l F_{l,j},$$

where the g_l are rational functions which are independent of j. It follows that

$$\begin{aligned} \det[F_{i,j}] &= \det \left[h^{i-1} \frac{d^{i-1} F_{1,j}}{dt^{i-1}} \right] \\ &= h^{(s-1)s/2} \det \left[\frac{d^{i-1} F_{1,j}}{dt^{i-1}} \right] . \end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{ord}_P \rho &\geqq \operatorname{ord}_P h^{(s-1)s/2} \\ &= \dim \tilde{\mathcal{O}}/(h^{(s-1)s/2}) \\ &= \frac{(s-1)s}{2} \dim \tilde{\mathcal{O}}/(h) \\ &= (s-1)\cdot s \cdot \delta_P,\end{aligned}$$

with the final equality coming from the fact that X is Gorenstein at P. ■

COROLLARY 1. *If P is a singular point of X and $\mathcal{L}$ is an invertible sheaf on X such that $h^0(\mathcal{L}) > 1$, then P is a Weierstrass point of $\mathcal{L}$.*

The notion of gaps does not appear to extend to singular points. If P is a singular point, then one is interested not in the (Weil) divisors nP, but rather in all (0-dimensional) subschemes supported at P. As one of his main results, Widland proved:

THEOREM 1. *Suppose that X is an integral, projective Gorenstein curve of arithmetic genus $g > 1$ and suppose $P \in X$. Then the following are equivalent.*

(1) $W_\omega(P) > 0$.

(2) *There is a nonzero $\sigma \in H^0(X, \omega)$ satisfying $\operatorname{ord}_P \sigma \geqq g$.*

(3) *There is a 1-special subscheme (cf. [4]) with support P and length equal to g.*

(4) *There is a 1-special subscheme with support P and length at most g.*

EXAMPLE. Let X denote the rational nodal curve obtained from $\mathbf{P}^1$ by identifying 0 with ∞, 1 with -1, and i with $-i$. Then Widland showed that ω_X has no nonsingular Weierstrass points and that at a node P there are no *principal* 1-special subschemes of length at most 3. On a generic rational nodal curve with g nodes the canonical bundle has $g(g-1)$ nonsingular Weierstrass points [7].

3. A rational nodal curve with a non-dense set of W-pts.

In this section, X will denote an irreducible, projective rational curve with $g > 1$ nodes. X may be obtained by identifying g pairs of distinct points $\{b_j, c_j\}$, $j = 1, \ldots, g$, of $\mathbf{P}^1$. We assume that none of these $2g$ points is ∞. We recall briefly the definition and chief properties of the generalized Jacobian of X. Our main reference for this material is [9].

Pic(X) is defined to be the group of divisors

$$D = \sum n_k x_k, \qquad x_k \in \mathbf{P}^1 - \{b_1, c_1, \ldots, b_g, c_g\},$$

modulo the equivalence relation given by $D \equiv 0$ if $D = (f)$, where f is a rational function on $\mathbf{P}^1$ such that $f(b_j) = f(c_j)$ for all $j = 1, \ldots, g$. Then $J = \mathrm{Jac}(X)$ is the group Pic^0 corresponding to divisors of degree 0. We have a group isomorphism $J \cong (\mathbf{C}^*)^g$ given as follows. If D is a divisor of degree 0, then as a divisor on $\mathbf{P}^1$ it equals the divisor of zeros and poles of a rational function f. The isomorphism above is then

$$\iota \colon D \mapsto \left(\frac{f(b_1)}{f(c_1)}, \ldots, \frac{f(b_g)}{f(c_g)}\right).$$

We will identify J with $(\mathbf{C}^*)^g$ via this isomorphism. Accordingly, we will write the operation in J as multiplication, although we will still write the operation in the group of divisors as addition. Fix a smooth point $x_0 \in X$ and let X_0 denote the set of smooth points of X. Then we have a map

$$\varphi \colon \left\{D = \sum n_k x_k \colon x_k \in X_0\right\} \to J$$

given by

$$D \mapsto \iota(D - \deg(D) \cdot x_0).$$

We then have a map $X_0 \to J$ given by $x \mapsto \varphi(x - x_0)$. Take $x_0 = \infty$. Then this map is just

$$\varphi \colon x \mapsto \left(\frac{b_1 - x}{c_1 - x}, \ldots, \frac{b_g - x}{c_g - x}\right).$$

(Here, we make the convention that $(b_j - \infty)/(c_j - \infty) = 1$.) This map extends to give the "Abel–Jacobi" mapping from effective divisors of degree m with support in the smooth locus to the Jacobian:

$$X_0^{(m)} \xrightarrow{\varphi} J,$$

$$\sum n_k x_k \mapsto \left(\Pi_k \left(\frac{b_1 - x_k}{c_1 - x_k}\right)^{n_k}, \ldots, \Pi_k \left(\frac{b_g - x_k}{c_g - x_k}\right)^{n_k}\right).$$

Following Mumford, define a function τ_X on J by

$$\tau_X(\lambda_1, \ldots, \lambda_g) = \det \begin{bmatrix} 1 - \lambda_1 & \cdots & 1 - \lambda_g \\ b_1 - c_1\lambda_1 & \cdots & b_g - c_g\lambda_g \\ \vdots & & \vdots \\ b_1^{g-1} - c_1^{g-1}\lambda_1 & \cdots & b_g^{g-1} - c_g^{g-1}\lambda_g \end{bmatrix}$$

According to Mumford [9, p. 3.251], "this determinant is the analog of ϑ and its zeros ... are the analog of θ [the theta divisor]." We will let Θ denote the zero set of τ_X.

LEMMA 1. *Suppose $x_1, \ldots, x_g \in X_0$. Then $x_1 + \cdots + x_g - \infty$ is effective if and only if $\tau(\varphi(x_1 + \cdots + x_g)) = 0$.*

PROOF. [9, p. 3.251].

COROLLARY 2. $\tau(\varphi(X_0^{(g-1)})) = 0$.

PROOF. Suppose $x_1 + \cdots + x_{g-1} \in X_0^{(g-1)}$. Then $x_1 + \cdots + x_{g-1} + \infty - \infty$ is effective. Hence,

$$\tau(\varphi(x_1 + \cdots + x_{g-1})) = \tau(\varphi(x_1 + \cdots + x_{g-1} + \infty)) = 0. \qquad \blacksquare$$

The following lemma is similar to a lemma of Olsen [12, p. 362], but in a very special case.

LEMMA 2. *Suppose $g = 2$. Suppose $\mathscr{L}$ is an invertible sheaf on X such that $h^1(\mathscr{L}) = 0$. Put $s = h^0(\mathscr{L})$ and suppose $s > 0$. If P is a smooth W-pt. of $\mathscr{L}$, then*

$$\varphi(\mathscr{L}) \cdot (\varphi(P))^{-s} \in \Theta.$$

(Here, $\varphi(\mathscr{L})$ denotes the image of the divisor of any global section of $\mathscr{L}$ with support contained in X_0 under the Abel–Jacobi map.)

PROOF. Since P is a smooth W-pt. of $\mathscr{L}$, there exists a nonzero section σ in $H^0(X, \mathscr{L}(-sP))$. Then $\operatorname{div}(\sigma) - sP = D$ is an effective Cartier divisor of degree one by the Riemann–Roch Theorem for Gorenstein curves. But then we must have $D = Q$ for some $Q \in X_0$, since an effective Cartier divisor of degree one cannot have support in the singular locus. Hence,

$$\varphi(\operatorname{div}(\sigma) - sP) = \varphi(\mathscr{L}) \cdot \varphi(sP)^{-1} = \varphi(\mathscr{L}) \cdot \varphi(P)^{-s} = \varphi(Q) \in \Theta. \qquad \blacksquare$$

Now put X equal to the irreducible rational curve obtained from $\mathbf{P}^1$ by identifying -3 with 0 and 1 with -1. Then

$$\tau = \tau_X(\lambda_1, \lambda_2) = \lambda_1\lambda_2 + \lambda_1 - 4\lambda_2 + 2.$$

Put

$$V = \{(\lambda_1, \lambda_2) \in J : |\lambda_1| < 1 \text{ and } |\lambda_2| < \tfrac{1}{5}\}.$$

LEMMA 3. $V \cap \Theta = \varnothing$.

PROOF. We consider the image of the unit circle $|z|=1$ under the linear fractional transformation

$$w=T(z)=\frac{-z-2}{z-4}.$$

It is easy to see that this image is the circle

$$|w-\tfrac{3}{5}|=\tfrac{2}{5}$$

and that the interior of $|z|=1$ goes into the interior of this circle. Hence, if $|z|<1$ then $|T(z)|>\frac{1}{5}$. It follows that if $|\lambda_1|<1$ and $(\lambda_1,\lambda_2)\in\Theta$, then $|\lambda_2|>\frac{1}{5}$. ∎

Let k be any nonzero complex number and put

$$f(z)=z^4+3z^3-z^2-3z+k.$$

Note that $f(-3)=f(0)=f(1)=f(-1)=k$. Let $\mathscr{L}$ denote the invertible sheaf on X corresponding to the divisor of zeros of $f(z)$. Since $\deg(\mathscr{L})=4>2g-2$, we have:

LEMMA 4. *$h^1(\mathscr{L})=0$, $h^0(\mathscr{L})=3$ and $\varphi(\mathscr{L})=(1,1)$.*

Let U denote the interior of the circle

$$|z-\tfrac{5}{6}|=\tfrac{1}{6}.$$

We may view U as an open subset of X_0.

LEMMA 5. *$\varphi(U)^{-1}\subseteq V$.*

PROOF. If $x\in X_0$, then

$$\varphi(x)^{-1}=\left(\frac{x}{x+3},\frac{x-1}{x+1}\right).$$

Suppose $x\in U$ and put $(\lambda_1,\lambda_2)=\varphi(x)$. Since $|x|<|x+3|$, we have that $|\lambda_1|<1$. Since the image of the interior of the circle

$$|z-\tfrac{5}{6}|=\tfrac{1}{6}$$

under the linear fractional transformation

$$w=\frac{z-1}{z+1}$$

is the interior of the circle

$$|w + \tfrac{1}{10}| = \tfrac{1}{10},$$

it follows that $|\lambda_2| < \frac{1}{5}$. ∎

Put

$$W = \{P \in X\colon P \text{ is a Weierstrass point of } \mathscr{L} \text{ of order } n \text{ for some } n \geqq 1\}.$$

THEOREM 2. $U \cap W = \varnothing$.

PROOF. By Lemma 2, if $P \in X_0$ is a Weierstrass point of $\mathscr{L}^{\otimes n}$, then

$$\varphi(\mathscr{L})^n \cdot \varphi(P)^{-s_n} = \varphi(P)^{-s_n} \in \Theta,$$

where $s_n = h^0(\mathscr{L}^{\otimes n})$. But, if $P \in U$, then $\varphi(P)^{-1} \in V$. Therefore, $\varphi(P)^{-m} \in V$ and $\varphi(P)^{-m} \notin \Theta$ for all $m \geqq 1$. Hence, there are no higher order Weierstrass points of $\mathscr{L}$ in U. ∎

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